I. INTRODUCTION

Group codes provide the possibility to use more spectrally efficient signal constellations while keeping many good qualities of binary-linear codes [1]. Also for channels with certain symmetric properties, like MPSK-AWGN channels, codes over algebraic structures with weaker algebraic structure than fields have better properties [2]. The concept of group code over a group $G$, that will be used in this work, is in the sense of [1], [2], [3], that is, a group code $C$ is a subgroup of $G^N$, where $G^N = \widetilde{G} \oplus G \oplus \cdots \oplus G$ with $\oplus$ representing the direct product of groups. Equivalently the group code $C$ is the image of an encoder mapping

$$\phi : U \rightarrow G^N,$$

that must be an injective group homomorphism. Then the encoded set $U$ is also a group and isomorphic with $C$.

In this paper we propose a constructive definition of encoding capacity of channels $(X, Y, p(y|x))$ having its input alphabet $X$ bijectively matched with an extension group $G = \mathbb{Z}_{2^p} \oplus \mathbb{Z}_{2^q}$. Following [1], from which is adapted this definition, this encoding capacity is also called $G$-capacity and it is denoted by $C_G$. Roughly speaking the formula to calculate $C_G$ is a minimal choice among weighted capacities, in the Shannon sense, of the sub-channels induced by the sub-groups of $G$. The channel capacity $C = \max_{p(y|x)} H(Y) - H(Y|X)$, with $p(y|x)$ being the probability density function of the noise, is one of these sub-channels. When $C = C_G$ then it is said that the encoding capacity achieves the channel capacity.

To give examples of application we use two symmetric channels. The first is a three-dimensional channel with group code over the dihedral group of 8 elements. The input alphabet of this channel is a parameter dependent signal set. It is shown that for some values of the parameter the channel capacity is not achieved. The second example is a four-dimensional symmetric channel with group code over the quaternions group of 8 elements.

II. DIRECT PRODUCT POWER OF EXTENSION OF GROUPS

A group $G$ with normal subgroup $H \triangleleft G$ such that the quotient group $G/H$ is isomorphic with a group $K$ is said to be an extension of $H$ by $K$ [4]. Since each element $g \in G$ is in a unique lateral class $Hk \in G/H$ then $g$ can be written as a “ordered pair” $g = hk$. This determines a group isomorphism between $G$ and $H \times K$. The semi-direct product and direct product of groups are particular cases of extension of groups. In this article the extension $H \times K$ will be represented by the symbol “box-times”: $H \boxtimes K$. The group operation on these pairs are performed with the rule $g_1g_2 = (h_1k_1)(h_2k_2) = h_1(k_1h_2k_1^{-1})k_1k_2$, where $k_1h_2k_1^{-1}$, denoted in the literature about algebra as $hk^k_2$, is in $H$ and $k_1k_2 \in K$. It can be shown that when $k_1^k_2 \neq h_2$, for some $h_2$ or else some $k_1$, then $G \cong H \boxtimes K$ is a non-Abelian group.

**Proposition 1:** For an integer $N \geq 1$, if $G = H \boxtimes K$ then

$$G^N = (H \boxtimes K)^N \cong H^N \boxtimes K^N.$$
that \( \varphi \) is a surjective group homomorphism with kernel \( H^N \). Therefore \( H^N \) is a normal subgroup of \( G^N = (H \boxtimes K)^N \) and \( G^N/H^N \cong (G/H)^N \cong K^N \).

Some important finite non-Abelian groups are extensions \( H \boxtimes K \), where both \( H \) and \( K \) are Abelian or else cyclic. For example, the dihedral group \( D_n \) is an extension \( D_n = \mathbb{Z}_n \boxtimes \mathbb{Z}_2 \), where \( \mathbb{Z}_n \) is the cyclic group \( \{0, 1, \ldots, n-1\} \). The generalized quaternion \( Q_{2^n} \) is also an extension \( Q = \mathbb{Z}_{2^{n-1}} \boxtimes \mathbb{Z}_2 \). Also the alternating group \( A_n \), which is non-abelian and has 12 elements is an extension \( \mathbb{Z}_{2} \boxtimes \mathbb{Z}_3 \). These and other extensions have representations in the families of orthogonal matrices \( O(2, \mathbb{R}) \), \( O(3, \mathbb{R}) \), \( O(4, \mathbb{R}) \cong O(2, \mathbb{C}) \), where \( \mathbb{R} \) is the real field and \( \mathbb{C} \) is the complex field. So, this matrix representation possibility together with the distribution of the exponent \( (H \boxtimes K)^N \cong H^N \boxtimes K^N \) makes that group extension may be very suitable for group codes over channels whose input alphabet is matched to \( G \).

### III. Encoding Capacity of Channels with Group Codes over Extensions

In this section we study group codes over \( G \) and channels \((X, Y, p(y|x))\) such that: 1) \( G \) and \( X \) are one-to-one matched and 2) \( G \) is an extension \( G = \mathbb{Z}_{p_1^n} \boxtimes \mathbb{Z}_{p_2^m} \), where \( p_1 \) and \( p_2 \) are prime numbers that do not need to be different.

If \( G = \mathbb{Z}_{p_1^n} \boxtimes \mathbb{Z}_{p_2^m} \), by the Proposition 1, \( G^N \) must have the form:

\[
G^N = \mathbb{Z}_{p_1^n}^N \boxtimes \mathbb{Z}_{p_2^m}^N.
\]

From here, the group code \( C \) or else the uncoded group \( U \) must have the structure:

\[
U = \left( \mathbb{Z}_{p_1^{k_1}} \boxtimes \mathbb{Z}_{p_1^{k_2}} \boxtimes \cdots \boxtimes \mathbb{Z}_{p_1^{k_r}} \right) \boxtimes \mathbb{Z}_{p_2^{k_2}}.
\]

where \( k_1 + k_2 + \cdots + k_r \leq N/2 \) and \( k_{21} \leq N \).

Then each subgroup \( U \) of \( G^N \) is determined by the array \( k = (k_1, k_2, \ldots, k_r) \).

Let \((X, Y, p(y|x))\) be the channel with \( X \) one-to-one matched with \( G \). Then, as it was said before, the channel can be represented by \((G, Y, p(y|g))\). The subgroups of \( G \) will induce sub-channels that will have their respective subgroup codes of \( U \). To show how are these subgroup codes over these sub-channels it will be used arrays of integers \( l = (l_{11}, l_{12}, \ldots, l_{1r}) \) such that \( l_{ij} \leq j \) for all \( i, j \). Then, let \( U(l) \) and \( G(l) \) be groups defined by the following formulas:

\[
U(l) = \bigoplus_{i=1}^r p_1^{l_{i1}} \boxtimes p_1^{l_{i2}} \boxtimes \cdots \boxtimes p_1^{l_{ir}} \boxtimes \mathbb{Z}_{p_2^{k_2}}
\]

and

\[
G(l) = \left[ \sum_{j=1}^r p_1^{-l_{i1}} H(p_1) \right] \boxtimes p_2^{l_{i2}} \boxtimes \mathbb{Z}_{p_2^m},
\]

where \( H = \mathbb{Z}_{p_1^n} \) and \( H(p_1) \) is the subgroup of elements of \( H \) with order \( p_1 \).

Since each \( p_1^{-l_{i1}} \mathbb{Z}_{p_1^n}^{k_{i1}} \sim \mathbb{Z}_{p_1^n}^{k_{i1}} \), then \( U(l) \cong \bigoplus_{j=1}^r \mathbb{Z}_{p_1^n}^{k_{i1}} \boxtimes \mathbb{Z}_{p_2^{k_2}} \) which shows that \( U(l) \) is a subgroup of \( U \). Moreover if \( l_{ij} = j \) for all \( i, j \) then \( U(l) = U \). On the other hand, for \( G(l) \), we have that \( H \left( p_2 \right) = p_1^{l_{i1}} \mathbb{Z}_{p_1^n} \), then \( p_1^{-l_{i1}} H(p_1) = p_1^{-l_{i1}} \mathbb{Z}_{p_1^n} \). Hence \( \sum_{j=1}^r p_1^{-l_{i1}} H(p_1) \cong \mathbb{Z}_{p_1^n}^{k_{i1}} \). Where \( l_{1m} = \max \{ l_{11}, l_{12}, \ldots, l_{1r} \} \). With this \( G(l) \cong \mathbb{Z}_{p_1^n}^{k_{i1}} \boxtimes \mathbb{Z}_{p_2^{k_2}} \) which shows that it is a subgroup of \( G \). Therefore, \( U(l) \) is a subgroup code over the sub-channel \((G(l), Y, p(y|g))\), that is:

\[
U(l) \subset G(l)^N.
\]

The encoding rates of \( U(l) \) and \( U \) can be calculated by

\[
R_l = \frac{\log(|U(l)|)}{N} = \frac{1}{N} \sum_{i=1}^r \sum_{j=1}^2 l_{ij} k_{ij} \log(p_i) \tag{6}
\]

and

\[
R = \frac{1}{N} \sum_{i=1}^r \sum_{j=1}^2 j k_{ij} \log(p_i), \tag{7}
\]

where \( r_1 = r \) and \( r_2 = 1 \). If \( \alpha_{ij} = \frac{j k_{ij} \log(p_i)}{1} \) then \( \sum_{i,j} \alpha_{ij} = 1 \) and \( k_{ij} = \alpha_{ij} \frac{\log(|U|)}{1} \).

Thus,

\[
R_l = R \sum_{i,j} l_{ij} \alpha_{ij}.
\]

Let \( C_l \) be the capacity of \((G(l), Y, p(y|g))\), then

\[
R \sum_{i,j} \alpha_{ij} = R_l \leq C_l.
\]

Therefore:

\[
R \leq \min_l \left\{ \frac{C_l}{\sum_{i,j} \alpha_{ij}} \right\}. \tag{8}
\]

Finally considering the family of probability arrays \( (\alpha_{ij}) \) where \( i = 1, 2 \) and \( j = 1, 2, \ldots, r \), such that \( \sum_{i,j} \alpha_{ij} = 1 \), we make the adaptation, for the extension group \( \mathbb{Z}_{p_1^n}^f \boxtimes \mathbb{Z}_{p_2^m} \), case, of the Definition 20 of [1]:

**Definition 1:** For the extension \( G = \mathbb{Z}_{p_1^n}^f \boxtimes \mathbb{Z}_{p_2^m} \), the G-encoding capacity of the G-symmetric channel \((G, Y, p(y|x), x \in G)\), is:

\[
C_G = \max_{(\alpha_{ij})} \min_l \left\{ \frac{C_l}{\sum_{i,j} \alpha_{ij}} \right\}. \tag{9}
\]

Let \( C \) be the capacity of the channel, that is, \( C \) is the maximal mutual information \( H(Y) - H(Y|X) \). For the maximal case of \( l \) where \( l_{ij} = j \), for all \( i, j \), we have \( G_l = G \) and the respective
capacities are the same: \( C_t = C \). Hence \( \min_t \left\{ \frac{C_t}{\sum_{j} \rho_{ij}} \right\} \leq C \).

Therefore:
\[
C_G \leq C. \tag{10}
\]

When \( C_G = C \) then it is said that the \( G \)-encoding capacity of the
\( G \)-symmetric channel achieves the capacity of the Channel [1].

**Lemma 1:** Let \( \rho^0 \) be the array
\[
\left( \frac{l_{21}}{\rho_{ij}} \right)
\]
for some \( 1 \leq \rho \leq r \).

Then
\[
\frac{C_{\rho}}{\sum_{j} \rho_{ij}} \leq \frac{C_{l}}{\sum_{j} \frac{l_{ij}}{l_{ij}}} \quad \text{for each array } l = (l_{ij}).
\]

max\{\( l_{11}, l_{12}, \ldots, l_{1r} \)\} = \( \rho \).

**Proof:** Since each subgroup \( \rho_{ij} \) from (4) is equal to \( \rho_{ij} \) when \( G(\rho^0) = G(l) \). Thus the capacities of the sub-
channels determined by these subgroups must be also equal,
that is, \( C_{\rho} \) and \( C_l \).

On the other hand,
\[
\sum_{i,j} \frac{\rho}{j} \alpha_{ij} = \sum_{j=1}^{\rho} \alpha_{ij} + \rho \sum_{j=\rho+1}^{r} \alpha_{ij} + \rho \sum_{j=\rho+1}^{r} \alpha_{ij} \leq 0.
\]

The Lemma 1 allow us to simplify the formula (9) to:
\[
C_G = \max_{\alpha_{ij}} \min_{\rho=1,2,\ldots,r} \left\{ \frac{C_{\rho}}{\sum_{j} \frac{\rho}{j} \alpha_{ij}} \right\}. \tag{11}
\]

### IV. TWO EXAMPLES OF NON-ABELIAN GROUP CODES OVER SYMMETRIC CHANNELS

Given a group \( G \) and a set \( \mathcal{X} \), it is said that \( G \) acts over
\( \mathcal{X} \) when a) \( g_1(g_2x) = g_1g_2(x) \) for all \( g_1, g_2 \in G \) and for all
\( x \in \mathcal{X} \). b) \( ex = x \), for all \( x \in \mathcal{X} \), e is the identity element of \( G \).

The action is **transitive** when for all \( x_1, x_2 \in \mathcal{X} \) there is such \( x \in \mathcal{X} \) such that \( x_2 = gx_1 \). Forney in [3] calls a signal set \( \mathcal{X} \) **geometrically uniform** when \( \mathcal{X} \) enjoy the transitive action of a group \( G \) of isometric matrices. The action of \( G \) on \( \mathcal{X} \) is said to be **simply transitive** if for all \( x_1, x_2 \in \mathcal{X} \) there is an unique \( g \in G \) such that \( x_2 = g\cdot x_1 \). Another type of action is the so called **isometric action**. For the case where \( \mathcal{X} \) is a continuous subset of \( \mathbb{R}^n \), it is said that \( G \) acts isometrically on \( \mathcal{X} \) when it preserves Euclidean distances, that is, \( ||x|| = ||g\cdot x|| \), for all \( g \in G \) and for all \( x \in \mathcal{X} \). For the case where \( \mathcal{X} \) is a finite set, any group action is isometric action [1].

**Definition 2:** Let \( G \) be a group. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be sets with
joint probability distribution \( p_{XY}(x,y) \) and conditional probability
distribution \( p_{Y|X}(y|x) \) denoted as \( p(y|x) \). A memoryless channel \( (\mathcal{X}, \mathcal{Y}, p(y|x)) \) is said to be \( G \)-symmetric if
- \( G \) acts simply transisitively on \( \mathcal{X} \),
- \( G \) acts isometrically on \( \mathcal{Y} \),
- \( p(y|x) = p(y|gx) \) for all \( g \in G \), for all \( x \in \mathcal{X} \), for all \( y \in \mathcal{Y} \). [1]

The simply transitive action of \( G \) over \( \mathcal{X} \) implies that \( G \) and \( \mathcal{X} \) are one-to-one matched, thus we can use the formulas (9)
or else its simplified version (11) to compute the \( G \)-capacity of
\( G \)-symmetric channels.

### A. The dihedral case 3D

The group of symmetries of the square \( D_4 \) is a non-
Abelian group that is an extension \( \mathbb{Z}_4 \rtimes \mathbb{Z}_2 \) where \( \mathbb{Z}_4 = \{a, a^2, a^3, e\} \) and \( \mathbb{Z}_2 = \{b, e\} \). The generators \( a \) of \( \mathbb{Z}_4 \) and \( b \) of \( \mathbb{Z}_2 \) also generate \( D_4 \) with the group operation given by \( (a^{k_1}b^{k_2}) \cdot (a^{k_3}b^{k_4}) = a^{k_1+k_3}b^{k_2+k_4} \). For instance \( (ab)^2 = a^2b^2 \). This finite group has a representation in \( \text{O}(3, \mathbb{R}) \), the set of orthogonal matrices of the space \( \mathbb{R}^3 \), via the mapping \( a \mapsto \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) and \( b \mapsto \begin{pmatrix} \sqrt{2} & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \). With this representation is easy to verify that \( D_4 \) acts simply transisitively on the tri-dimensional signal set \( \mathcal{X}_8^3 \) defined by:

\[
\mathcal{X}_8^3 = \left\{ x_k = (0,1,2,\ldots,7), 0 \leq \beta < \infty; \quad x = \sqrt{-1} \right\}. \tag{12}
\]

Alternatively, this constellation also can be described in terms of
spherical coordinates as:

\[
\mathcal{X}_8^3 = \left\{ (\cos \varphi_k \cos \theta, \sin \varphi_k \cos \theta, (-1)^k \sin \theta) \right\}
\]

\[
\varphi_k = \frac{k \pi}{8}; \quad k = 0, 1, 2, \ldots, 7; \quad 0 \leq \theta < \frac{\pi}{2} \tag{13}
\]

where \( \theta \) = arctan(\beta). The Fig. 1 shows this constellation for
the case \( \beta = 1 \) or else \( \theta = \pi/4 \). For the extreme case \( \beta = 0 \) the
3-D constellation \( \mathcal{X}_8^3 \) turns into the 8PSK constellation on the
\( XY \)-plane. On the other side, when \( \beta \rightarrow \infty \) the constellation
\( \mathcal{X}_8^3 \) approaches to \( \{0, 0, 1\}, \{0, 0, -1\} \). If the signal set \( \mathcal{X}_8^3 \)
is transmitted over an AWGN channel where the noise has
probability density \( p(y) = \frac{1}{2\pi} e^{-\frac{1}{2} y^2}, y \in \mathbb{R}^3 \), then the conditional probability transitions of the channel are

\[
p(y|x_k) = \frac{1}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(-\frac{\|y - x_k\|^2}{2\sigma^2}\right)
\]

Using again the matrix representation of \( D_4 \) it can be shown that
\[
p(\gamma y|x_k) = p(y|x_k) \quad \text{for all} \quad y \in D_4, \quad x_k \in X_8^0, \quad \gamma \in \mathbb{R}^3.
\]

This is the entry of the random variable \( Y \) of the output of the sub-channel with probability density \( \lambda_{i,j,k}(y), y \in \mathbb{R}^3 \), and

\[
H(p_1) = H(Y|X = x_0) = -\int_{\mathbb{R}^3} p(y|x_0) \log(p(y|x_0))dy = \log(3\sqrt{2\pi\sigma e}).
\]

Thus we can write

\[
C_{i,j,k} = H(\lambda_{i,j,k}) - 3\log(\sqrt{2\pi\sigma e}). \tag{16}
\]

All the probability density functions \( \lambda_{i,j,k} \) to compute the capacities of the sub-channels are showed in the Table II

<table>
<thead>
<tr>
<th>Sub-group ( G(l_{i,j,k}) )</th>
<th>Sub-Constellation ( X(l_{i,j,k}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2Z_4 \otimes {0} = {e, a^2} )</td>
<td>( {x_0, x_1} )</td>
</tr>
<tr>
<td>( 2Z_4 \otimes {e} = {e, a, a^2} )</td>
<td>( {x_0, x_1, x_4, x_5} )</td>
</tr>
</tbody>
</table>

TABLE I. \( l_{i,j,k} \) ARRAYS, \( G(l_{i,j,k}) \) SUBGROUPS AND \( X(l_{i,j,k}) \) SUB-CONSTELLATIONS FOR THE \( D_4 \)-SYMMETRIC CHANNEL

The implementation of the formula (16) with the \texttt{triplequad} command of the software Octave [6] to compute the capacities of (15), shows that the achievement of the channel capacity depends on \( \beta \). For some \( \beta_0 \) such that \( 0.32 < \beta_0 < 0.72 \), if \( \beta > \beta_0 \) then the channel capacity is achieved, on the contrary, if \( \beta < \beta_0 \) then the channel capacity is not achieved. Some results, for fixed noise level \( \sigma = 0.5 \), of these computations are shown in the Table III. For instance, for \( \beta = 1 \) the channel capacity is not achieved: \( 3\log(\beta_{i,j,k}) - 2H(\lambda) = 2.6603 < H(p_1) = 3.1423 \) that means \( \frac{3\log(\beta_{i,j,k})}{2} < C \). It is interesting notice the behavior of the entropies as \( \beta \rightarrow +\infty \). \( H(\lambda_{i,j,k}) \rightarrow H(\lambda) \), whereas \( H(\lambda_{i,j,k}) \rightarrow 0 \) on the other hand for \( \beta = 0 \) the \( X_8 \) signal set becomes the 8PSK constellation. In [7], by using the \texttt{dblequad} command of Octave, it was shown that the 8PSK-AWGN channel with group code over \( D_4 \) achieves the channel capacity.

\[
\begin{array}{c|c|c|c|c|c}
\beta & 0.32 & 0.72 & 1.00 & 1.3 & 3.07 \\
\hline
H(\lambda_{i,j,k}) & H(\lambda) & 4.7709 & 4.8902 & 4.8323 & 4.6903 & 4.2371 \\
H(\lambda_{i,j,k}) & H(\lambda_{i,j,k}) & 4.8004 & 4.2875 & 4.1084 & 4.8831 & 3.3825 \\
H(\lambda_{i,j,k}) & H(\lambda_{i,j,k}) & 4.4523 & 4.6046 & 4.6113 & 4.5498 & 2.2545 \\
H(\lambda_{i,j,k}) & H(\lambda_{i,j,k}) & 4.7014 & 3.9422 & 3.8590 & 3.3605 & 3.6301 \\
\end{array}
\]

TABLE III. \( H(\lambda_{i,j,k}) \) ENTRIES OF THE OUTPUT RV \( \lambda(\lambda_{i,j,k}) \) \( D_4 \)-SYMMETRIC CHANNEL SUB-CHANNELS OF \( X_8 \) AND FIXED NOISE LEVEL \( \sigma = 0.5 \)

B. The quaternions case 4D

The group of quaternions \( Q_8 \) is also a non-Abelian group that can be expressed as an extension \( Z_4 \otimes Z_2 \). If \( a \) is the generator of \( Z_4 \) and \( b \) is the generator of \( Z_2 \), then the group operation of \( Q_8 \) is given by \( (a^kB^k) \ast (a^hB^k) = a^{h+k}b^{h+k} \), where \( k \) is the generator of \( Z_2 \). In this way, the complete list of elements is \( \{e, a, a^2, ab, a^2b, a^2b \} \). This finite group has a representation in \( O(2, \mathbb{C}) \cong O(4, \mathbb{R}) \) via the mapping
where $a \mapsto \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}$, where $j = \sqrt{-1} \in \mathbb{C}$, and $b \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Choosing the initial point $x_0 = (1, 0) \in \mathbb{C}^2$ we have that $ax_0 = (j, 0)$, $bx_0 = (0, 1)$ and so on. The complete matching list determined by the simply transitive action of $Q_8$ over $X$ is shown in the table IV.

If the signal set $X$ is transmitted over an AWGN channel where the noise has the probability density $p(y) = \frac{1}{2\pi \sigma^2} e^{-\frac{(y-x)^2}{2\sigma^2}}$, $y \in \mathbb{C}^2 \cong \mathbb{R}^4$, then the conditional probability transitions of the channel are

$$p(y|x) = \frac{1}{4\pi \sigma^4} \exp \left( -\frac{||y-x||^2}{2\sigma^2} \right).$$

Clearly $p(y|x_k) = p(x_k | y)$.

Applying the formula (4) for $l_{111} = \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right)$ we have

$$G(l_{111}) = (2\mathbb{Z}_2 \oplus 2\mathbb{Z}_4) \mathbb{Z}_2 = 2\mathbb{Z}_4 \mathbb{Z}_2 = \{e, a^2\} \mathbb{Z}_2 = \{e, b, a^2, a^2b\}.$$

The sub-constellation matched to this subgroup is $\{x_0, x_2, x_4, x_6\}$. As in for the $D_4$ case the subgroups and sub-constellations that allow the computation of the $Q_8$-capacity are organized in the Table V.

Then by same methodology applied in the $D_4$ case, it is obtained:

$$C \geq C_{Q_8} \geq \min \left\{ 3C_{\mathbb{Z}_4}, \frac{3C_{\mathbb{Z}_2}}{2}, \frac{3C_{\mathbb{Z}_{10}}}{2}, C \right\}.$$ 

Now $H(p_0) = 2 \log(2\pi e \sigma^2)$ and

$$C_{l_{ijk}} = H(\lambda_{l_{ijk}}) - 2 \log(2\pi e \sigma^2).$$

where the formulas for the densities $\lambda_{l_{ijk}}$ are organized in the Table VI.

V. CONCLUSIONS

We gave a definition of $G$-capacity for some non-Abelian groups which are extensions $G = \mathbb{Z}_{p_1}^n \mathbb{Z}_{p_2}$. This definition is an adaptation from the $G$-capacity for Abelian groups $H = \bigoplus_{i=1}^{s} \mathbb{Z}_{p_i}^{n_{ij}}$ given in [1]. To make this adaptation, it has been shown that if $G = H \mathbb{Z}$ then $G \cong H^N \mathbb{Z}$. We did not make an adaptation for extensions like:

$$G = \bigoplus_{i=1}^{s} \mathbb{Z}_{p_i}^{n_{ij}} \mathbb{Z}_{p'} \mathbb{Z}_{p'},$$

which would be a truly generalization of the Abelian case, because the analyzed examples $D_4$ and $Q_8$ did not require such a general formula, also because we do not found, for this general case, a simple proof showing that $H(I) \subset G(I)$ which is a critical fact to fit in the sub-codes $U(I)$ over the sub-channels $(G(I), \mathbb{Z}, p(x|y))$, $g \in G(I)$.

In the dihedral 3D example it was numerically shown that the channel capacity is not achieved. The same $X_{\mathbb{Z}_4}^3$ - AWGN channel with group code over the cyclic group $\mathbb{Z}_8$ was exhibited in [1] as an example where the $G$-capacity does not achieve the channel capacity. A possible explanation for this common behavior of this channel could be in the internal group structure of $\mathbb{Z}_8$ and $D_4$. Both $\mathbb{Z}_8$ and $D_4$ have a unique subgroup isomorphic to $\mathbb{Z}_2$, $2\mathbb{Z}_2 \subset \mathbb{Z}_8$ and $G(l_{120}) \subset D_4$. Whatever the sub-constellation matched to $2\mathbb{Z}_2$ it will be also matched to $G(l_{120})$. Therefore, the capacities of the respective sub-channels will be the same.

For the four-dimensional channel with group code over $Q_8$, remains as an unsolved problem whether the channel capacity is achieved or not.

REFERENCES


